Self-Consistent ρ Trajectory in the New Form of the Strip Approximation*

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The new form of the strip approximation, devised by Chew, is applied to the problem of "bootstrapping" a ρ trajectory in the π-π system. Even in the absence of other trajectories it is possible to obtain a self-consistent ρ trajectory and residue functions $\alpha(t)$ and $\gamma(t)$ for $t<0$, with strip widths in the range 150 to 300 m_π^2. A particular example is given in detail. The absence of the force from other trajectories, and the rapid variation of $\alpha(t)$ and $\gamma(t)$ for $t\geqslant 0$, mean that our results can not represent the real ρ trajectory, but at least they confirm the viability of the methods used.

I. INTRODUCTION

THIS paper is one of a series devoted to applying the new form of the strip approximation[1] to the calculation of the π-π scattering amplitude. The physical principles underlying the new form of the strip approximation have been given in previous papers.[1,2] The amplitude is represented by its dominant Regge poles, with singularities which satisfy the Mandelstam representation, and should be correct in the resonance region and in the region of Regge asymptotic behavior. If the principles of maximal analyticity of the first and second kinds are valid, it is hoped that with the physical Regge trajectories, such an amplitude will be self-consistent in the sense that the "potential" due to the crossed-channel singularities will generate the direct-channel singularities. Chew and Jones[2] have devised a set of equations which are suitable for investigating this possibility.

The problem has two parts, the calculation of the "potential," and the solution of the N/D equations in the presence of the logarithmic singularity which this potential exhibits. The singularity occurs at the point where the resonance region is matched to the Regge asymptotic region, the boundary of the strip. Some preliminary results of solving N/D equations with such a boundary condition have already been reported,[3] but only for a potential corresponding to the exchange of a fixed-spin particle. In this paper we report an attempt to "bootstrap" a complete trajectory. The full π-π amplitude has several trajectories, P, P', ρ, and probably others, and a search for self-consistency with so many parameters presents a formidable problem. Also the "potential" resulting from the exchange of even-signature trajectories has some curious features which are currently under investigation, but the ρ trajectory generates a potential which is very similar to the form obtained from a fixed-spin particle, and seems quite straightforward. The approximation of supposing that the ρ resonance alone dominates the π-π amplitude has often been made with fairly satisfactory results,[4] and so, as a preliminary to a more ambitious calculation, we have tried to find an amplitude in which the force from the ρ trajectory in the crossed channels generates an identical trajectory in the direct channel. This is not a true bootstrap situation, of course, because the potential also gives rise to an $I=0$ trajectory which has not been included in the input, but the fact that we have been successful in this more limited enterprise is somewhat encouraging.

In Sec. II we discuss the calculation of the potential following the prescription of Chew and Jones, and in succeeding sections we write down the N/D equations, and consider the parametrization of the residue and trajectory functions. The results presented in Sec. V show that it is indeed possible to obtain self-consistent ρ trajectories, $\alpha(t)$, and residues, $\gamma(t)$; or at least they are self-consistent for $t<0$. The output trajectories have a large imaginary part as Re α approaches 1, however, so the physical ρ can not be observed directly, but there is a peak in the cross section. Also the input ρ width is more than twice the experimental value. These facts, however, may only be an indication of the difficulty of continuing $\alpha(t)$ and $\gamma(t)$ into the region above threshold where they become complex, without a better representation of the double spectral function.

Finally we compare the results of this calculation with a formula used by Chew and Teplitz[5] in relating the π-π total cross section to the slope of the Pomeranchuk trajectory and the width of the ρ.

II. THE POTENTIAL FUNCTION

In the new form of the strip approximation the scattering amplitude is represented by a sum of six items from different regions of the double spectral functions,[6]

$$A(s,t)=\sum_i [R_i^{t_1}(s,t)+\xi_i R_i^{u_1}(s,u)] +\sum_j [R_j^{s_1}(t,s)+\xi_j R_j^{u_1}(t,u)] +\sum_k [R_k^{s_1}(u,s)+\xi_k R_k^{t_1}(u,t)], \quad (1)$$

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[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[2] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).
[3] D. C. Teplitz and V. L. Teplitz, Phys. Rev. **137**, B136 (1965).
[4] See for example, F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[5] G. F. Chew and V. L. Teplitz, Phys. Rev. **136**, B1154 (1964).
[6] Most of the formulas in this section are taken from Ref. 2, above, where their significance is explained.

where the summation is over the various leading trajectories and $\xi_{i,j,k}$ is the signature factor $(=\pm 1)$ of the trajectory in question,

$$R_i^{t_1}(s,t)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{R_i(t',s)}{t'-t}dt' \qquad (2)$$

$$=\tfrac{1}{2}\Gamma_i(s)\int_{t_1}^{\infty}\frac{P_{\alpha_i(s)}(-1-t'/2q_s^2)}{t'-t}dt', \qquad (3)$$

where this integral exists. For $\alpha>0$ we use the analytic continuation

$$R_i^{t_1}(s,t)=\tfrac{1}{2}\Gamma_i(s)\left\{-\frac{\pi}{\sin\pi\alpha_i(s)}P_{\alpha_i(s)}\left(1+\frac{t}{2q_s^2}\right)-\int_{-4q_s^2}^{t_1}\frac{P_{\alpha_i(s)}(-1-t/2q_s^2)}{t'-t}dt'\right\}. \qquad (4)$$

Here

$$\Gamma_i(s)=[2\alpha_i(s)+1]\gamma_i(s)(-q_s^2)^{\alpha_i(s)}. \qquad (5)$$

We define the reduced partial-wave amplitude for complex l by

$$B_l^{\pm}(s)=-\frac{1}{2\pi}\int_{-\infty}^{0}\frac{dt}{q_s^{2l+2}}\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]A^{\pm}(s,t), \qquad (6)$$

a form first given by Wong.[7] It has the advantage of only requiring a knowledge of $A(s,t)$ for $t<0$, where $\alpha(t)$ and $\gamma(t)$ are real. The $\pm$ correspond to even and odd signatures, respectively.

The left-hand cut function for a given partial wave in the s channel is then combining (V.4) and (V.5) of Ref. 2,

$$\begin{aligned}B_l^{P\pm}(s)=&-\frac{1}{2\pi}\int_{-\infty}^{0}\frac{dt}{q_s^{2l+2}}\left[ImQ_l\left(1+\frac{t}{2q_s^2}\right)\right]\\&\times\left\{\frac{1}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}[\sum_j R_j(s',t)\pm\sum_k R_k(s',t)]\right.\\&+\frac{1}{\pi}\int_{u_1}^{\infty}\frac{du'}{u'-u}\sum_j\xi_j[R_j(u',t)-R_j(u',t')]\\&\pm\frac{1}{\pi}\int_{t_1}^{\infty}\frac{du'}{u'-u}\sum_k\xi_k[R_k(u',t)-R_k(u',t')]\\&+\frac{1}{\pi}\int_{t_1}^{\infty}\frac{dt'}{t'-t}\sum_k\xi_kR_k(t',u')\\&\left.\pm\frac{1}{\pi}\int_{u_1}^{\infty}\frac{dt'}{t'-t}\sum_j\xi_jR_j(t',u')\right\}\\&+\frac{1}{\pi}\int_{-\infty}^{s_0-t_1}\frac{ds'}{s'-s}\sum_i\int_{s^0}^{s_0-s'}dt(1\pm\xi_i)R_i(t,s')\\&\times\frac{P_l(-1-t/2q_{s'}^2)}{-4(-q_{s'}^2)^{l+1}}. \end{aligned}\qquad (7)$$

In the π-π problem we have the complication of isotopic spin, but also symmetry in t and u. It is convenient to define

$$\Gamma^{II'}(t)=[2\alpha(t)+1]\gamma(t)(-q_t^2)^{\alpha(t)}\beta_{st}(I,I'), \qquad (8)$$

where $\beta_{st}(I,I')$ is some element of the usual crossing matrix from the t to s channels:

$$\beta_{st}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}. \qquad (9)$$

β_{su} differs by $(-1)^I$, so that only B_l^{P+} exists for even I, and only B_l^{P-} for $I=1$. Combining (4) and (7), we obtain, after some manipulation,

$$\begin{aligned}B_l^{P\pm}(s)=&\frac{1}{2\pi q_s^{2l+2}}\int_{-\infty}^{0'}dt\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]\\&\times\left\{\Gamma^{II'}(t)\int_{-4q_t^2}^{s_1}\frac{du'}{u'-s}P_{\alpha(t)}\left(-1-\frac{u'}{2q_t^2}\right)\right.\\&+\Gamma^{II'}(t)\xi\int_{-4q_t^2}^{s_1}\frac{du'}{u'-u}P_{\alpha(t)}\left(-1-\frac{u'}{2q_t^2}\right)\\&+\xi\int_{s_1}^{\infty}\Gamma^{II'}(t')P_{\alpha(t')}\left(-1-\frac{u'}{2q_{t'}^2}\right)\frac{du'}{u'-u}\\&\mp\xi\int_{s_1}^{\infty}\Gamma^{II'}(t')P_{\alpha(t')}\left(-1-\frac{u'}{2q_{t'}^2}\right)\frac{du'}{u'-t}\\&\textit{either }+\frac{\pi\Gamma^{II'}(t)}{\sin\pi\alpha(t)}\left[\xi P_{\alpha(t)}\left(-1-\frac{s}{2q_t^2}\right)\right.\\&\left.\left.+P_{\alpha(t)}\left(1+\frac{s}{2q_t^2}\right)\right]\right\}\quad\text{if}\quad\left(-1-\frac{s}{2q_t^2}\right)<1\\&\textit{or }+\Gamma^{II'}(t)\left[\pi P_{\alpha(t)}\left(-1-\frac{s}{2q_t^2}\right)\right.\\&\times\begin{Bmatrix}\cot\frac{1}{2}\pi\alpha(t) & \text{for } \xi=+1\\ -\tan\frac{1}{2}\pi\alpha(t) & \text{for } \xi=-1\end{Bmatrix}\\&\left.\left.-2Q_{\alpha(t)}\left(-1-\frac{s}{2q_t^2}\right)\right]\right\}\quad\text{if}\quad\left(-1-\frac{s}{2q_t^2}\right)>1\\&-(1\pm\xi)\frac{1}{8}\int_{-\infty}^{4-t_1}ds'\frac{\Gamma(s')}{(s'-s)(-q_{s'}^2)^{l+1}}\\&\times\int_{4-s'}^{t_1}dt'P_{\alpha(s')}\left(-1-\frac{t'}{2q_{s'}^2}\right)P_l\left(-1-\frac{t'}{2q_{s'}^2}\right).\end{aligned}\qquad (10)$$

We obtain such a contribution from each trajectory. We have made use of the fact that

$$P_\alpha(-z)=e^{\mp i\pi\alpha}P_\alpha(z)-(2/\pi)Q_\alpha(z)\sin\pi\alpha, \qquad (11)$$

[7] Private communication to G. F. Chew. See Ref. 2.

and note that

$$\mathrm{Im}[Q_l(z)]=\tfrac{1}{2}\pi P_l(z) \quad \text{for} \quad -1<z<+1$$
$$=-Q_l(-z)\sin\pi l \quad \text{for} \quad z<-1. \tag{12}$$

A FORTRAN program has been devised to calculate the function B_l^P for any input γ and α. To calculate both $B_l^{P+}(s)$ and $B_l^{P-}(s)$ from the exchange of a single trajectory, at a sufficient number of values of s to be able to solve the N/D equations [≈ 20], and with a sufficient number of values of l to be able to examine the output trajectories [≈ 10], requires about 6 min on an IBM 7094, if all the terms of Eq. (10) are included. It is found that the third and fourth terms of the right-hand side of Eq. (10), which involve $\Gamma(t)$ for $t>s_1$, and the final term of Eq. (10), which is the contribution of direct channel poles to the left-hand cut, are all very small for ρ exchange, and the results are not appreciably altered by neglecting them. In the results quoted in this paper these terms were neglected, but had they been included the curves of Figs. 1 to 4 would have been almost completely unchanged.

We also need to know $\mathrm{Im}B_l^P(s_1)$, and from the first term of Eq. (10) we find

$$\mathrm{Im}B_l^P(s_1)=\frac{1}{2\pi q_{s_1}^{2l+2}}\int_{-\infty}^{0} dt\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_{s_1}^2}\right)\right]$$
$$\times\pi\Gamma^{II'}(t)P_{\alpha(t)}\left(-1-\frac{s_1}{2q_t^2}\right). \tag{13}$$

III. THE N/D EQUATIONS

By representing the partial-wave amplitudes as[1]

$$A_l^{\pm}(s)=q_s^{2l}N_l^{\pm}(s)/D_l^{\pm}(s), \tag{14}$$

where $N_l^{\pm}(s)$ has the left-hand cut of $A_l^{\pm}(s)$, and the right-hand cut for $s>s_1$, and $D_l^{\pm}(s)$ has the right-hand cut for $4<s<s_1$, we obtain the integral equation

$$N_l(s)=B_l^P(s)+\frac{1}{\pi}\int_{s_0}^{s_1} ds'\frac{B_l^P(s')-B_l^P(s)}{s'-s}$$
$$\times\rho_l(s')N_l(s'), \tag{15}$$

with

$$D_l(s)=1-\frac{1}{\pi}\int_{s_0}^{s_1} ds'\frac{\rho_l(s')N_l(s')}{s'-s}, \tag{16}$$

where

$$\rho_l(s)=\left(\frac{s-4}{s}\right)^{1/2}\left(\frac{s-4}{4}\right)^l. \tag{17}$$

However, Eq. (15) is not of the Fredholm type, because $B_l\ (s)$ is logarithmically divergent as $s\to s_1$ due to the first term of Eq. (10). In fact,[8]

$$B_l^P(s)\underset{s\to s_1}{\longrightarrow}\frac{1}{\pi}\,\mathrm{Im}B_l^P(s_1)\ln(s_1-s) \tag{18}$$

and

$$\sin^2\delta_l(s_1)=\rho_l(s_1)\,\mathrm{Im}B_l\ (s_1), \tag{19}$$

where δ_l is the phase shift.

A method of coping with this singularity by introducing a resolvant kernel has been discussed by Chew.[8] He shows that the solution of Eq. (15) can be written

$$N_l(s)=\int_{s_0}^{s_1} O_l(s,s')N_l^0(s')ds', \tag{20}$$

where $N_l^0(s')$ is the solution of

$$N_l^0(s)=B_l^P(s)+\int_{s_0}^{s_1} ds'\,K_l'(s,s')N_l^0(s'). \tag{21}$$

Expressions have been given[9] for $O_l(s,s')$ and $K_l'(s,s')$ in terms of $B_l^P(s)$, $\sin^2\delta_l(s_1)$, and s_1.

Apart from this complication, the determination of $N_l(s)$ and $D_l(s)$ from Eqs. (15) and (16) is straightforward. Details of a FORTRAN program for solving the equations are available.[10]

A pole in the amplitude is represented by a zero of the D function, and the trajectory of such poles is the function $\alpha(s)$ such that

$$D_{\alpha(s)}(s)=0. \tag{22}$$

Above threshold both D and α have imaginary parts, but if these are small it remains approximately true for all s that

$$\mathrm{Re}[D_{\mathrm{Re}[\alpha(s)]}(s)]=0. \tag{23}$$

For a Regge pole of the form

$$A(s,t)=\frac{\Gamma(s)P_{\alpha(s)}(1+t/2q_s^2)}{\sin\pi\alpha(s)}, \tag{24}$$

the t discontinuity is

$$A_t(s,t)=-\Gamma(s)P_{\alpha(s)}(-1-t/2q_s^2), \tag{25}$$

since

$$\mathrm{Im}[P_\alpha(z)]=-P_\alpha(-z)\sin\pi\alpha \quad \text{for} \quad z>1. \tag{26}$$

Thus the partial-wave projection of Eq. (24) is

$$B_l(s)=-\Gamma(s)\int_0^\infty P_{\alpha(s)}\left(-1-\frac{t}{2q_s^2}\right)Q_l\left(1+\frac{t}{2q_s^2}\right)$$
$$\times\frac{dt}{2(q_s^2)^{l+1}}, \tag{27}$$

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[8] G. F. Chew, Phys. Rev. **130**, 1264 (1963).
[9] V. L. Teplitz, Phys. Rev. **137**, B142 (1965).
[10] D. C. Teplitz and V. L. Teplitz, Lawrence Radiation Laboratory Report UCRL-11696, 1964 (unpublished).

and combining Eq. (11) with[11]

$$\int_1^\infty P_\alpha(z)Q_l(z)=\frac{1}{(l-\alpha)(l+\alpha+1)} \tag{28}$$

and

$$\int_1^\infty [Q_l(z)]^2=\frac{\psi'(l+1)}{2l+1}, \tag{29}$$

we find that if $\alpha(s_R)=l$, then

$$B_l(s) \underset{s\to s_R}{\longrightarrow} \frac{\Gamma(s_R)}{\alpha'(s_R)[s_R-s][2\alpha(s_R)+1](-q_{s_R}{}^2)^{\alpha(s_R)}}, \tag{30}$$

which, from Eq. (5),

$$=\gamma(s_R)/\alpha'(s_R)[s_R-s]. \tag{31}$$

Thus the residue of a pole of $B_l(s)$ is

$$\left[\frac{N_l(s)}{(dD_l(s)/ds)}\right]_{s=s_R}=\frac{\gamma(s_R)}{\alpha'(s_R)}. \tag{32}$$

With this expression we can obtain $\gamma(s)$ from the solution of the N/D equations.

IV. REGGE-POLE PARAMETERS

Unfortunately there is very little experimental knowledge of the Regge parameters to guide us in our choice of trial functions. Within the framework of this calculation we know that α and γ are real analytic functions cut from threshold to ∞, and so we can write[1]

$$\alpha(t)=\alpha_0+\int_{t_0}^\infty dt'\frac{\rho_\alpha(t')}{t'-t}, \tag{33}$$

$$\gamma(t)=\gamma_0+\int_{t_0}^\infty dt\frac{\rho_\gamma(t')}{t'-t}. \tag{34}$$

Very little is known about the forms of ρ_α and ρ_γ except that they must be small in the region where resonances occur. The strip approximation also requires that $\rho_{\alpha,\gamma}(t)$ be negligible for $t>t_1$, so the main weight of the ρ's must lie in the region between the highest resonances and the strip boundary. Since we only require α and γ for $t<0$, it is possible to make simple approximations to the integrals (33) and (34).

For α we take a three-parameter form,

$$\alpha(t)=\alpha_0+\alpha_1/(1-t/t_B), \tag{35}$$

however, if we also require $\alpha(28)=1$, corresponding to the ρ meson, we can reduce the parameters to two.

We take

$$\alpha(t)=1-\frac{at_B}{28}-a\left(1-\frac{t_B}{28}\right)\bigg/\left(1-\frac{t}{t_B}\right), \tag{36}$$

where $(1-a)$ is the intercept of the trajectory with $t=0$. It was found that a similar pole approximation was not suitable for the residue function, since the output would not reproduce such a behavior. Instead it was found convenient to make use of a formula given by Chew and Teplitz,[5]

$$\gamma(t)/\alpha'(t)\approx(\bar{t}-t)B_{\alpha(t)}{}^P(\bar{t}). \tag{37}$$

The difference between our function $B_l{}^P(t)$ and the function obtained from the exchange of an "elementary" (fixed spin) ρ is not great, and we can approximate

$$B_l(t)\approx\text{const}\left(1+\frac{t}{2q_\rho{}^2}\right)\frac{Q_l(1+m_\rho{}^2/2q_t{}^2)}{(q_\rho{}^2)^{l+1}} \tag{38}$$

and obtain

$$\gamma(t)=c\alpha'(t)[\bar{t}-t]\frac{Q_{\alpha(t)}[1+56/(\bar{t}-4)]}{[(\bar{t}-4)/4]^{\alpha(t)+1}}, \tag{39}$$

where c is some constant. This parametrization has the advantage of relating the parameters of γ to those of α, leaving only two further variables, c and $\bar{t}$. Also, γ is a slowly varying function of both $\bar{t}$ and t. Our program thus consists of varying the four parameters a, t_B, c, and $\bar{t}$ until self-consistency is achieved. From (31) the width of the ρ will be

$$[\gamma(m_\rho{}^2)/\alpha'(m_\rho{}^2)][q_\rho{}^2/m_\rho]=1.134\gamma(28)/\alpha'(28). \tag{40}$$

However, this involves the use of the functions above $t=0$, where we can no longer rely on them.

V. RESULTS

A search was made for a self-consistent set of parameters for the ρ with the various choices of s_1. We use $\beta^{11}=\frac{1}{2}$, $\xi=-1$, and solve the N/D equations for A^-. It was found that a flat input trajectory gave a steep output trajectory and vice versa, so it was fairly easy to make a search, varying a and t_B until both input and output trajectories had the same shape. The slope of the output trajectory is certainly not independent of the form of the input residue function, but with our parmetrization the main t dependence of (39) is in α, so that in choosing α we have approximately fixed γ except for the overall constant c. Other forms of residue function gave much less good results.

By adjusting c one can alter the height of the output trajectory until it coincides with the input. Finally $\bar{t}$ can be varied to try to make the input and output residues as similar as possible, though this requires a compensating adjustment of c.

There is no unique self-consistent solution. It is possible to obtain near self-consistency with various combinations of s_1 in the range 150 to 300 $m_\pi{}^2$, a from 0.2 to 0.35 (i.e., intercept of 0.8 to 0.65), and $\bar{t}$ from $\frac{1}{4}s_1$ to $\frac{3}{4}s_1$, taking c to be the dependent variable. In view of the large amount of computer time involved,

[11] Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Eqs. 3.12(4) and 3.12(6).

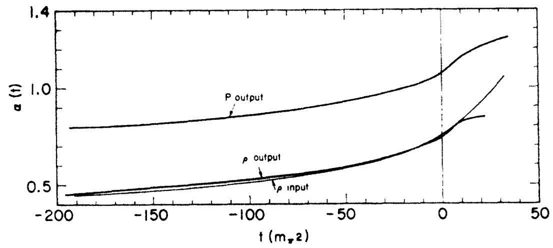

FIG. 1. The input and output trajectories $\alpha_{\rho\text{in}}=0.330+0.420/(1-t/75)$.

and the lack of correspondence to the real world through the neglect of the force from other trajectories, we did not carry out an exhaustive search, and quote here just one of our better results, without claiming that more perfect self-consistency, or a closer approximation to the physical ρ, can not be obtained.

With $s_1=200\ m_\pi^2$, it was found that there is good self-consistency if $a=0.25$, $t_B=75$, $c=107$, and $\bar{t}=60$. Thus

$$\alpha(t)=0.330+0.420/(1-t/75) \tag{41}$$

and

$$\gamma(t)=\frac{3370}{(75-t)^2}\frac{Q_{\alpha(t)}(2)}{(14)^{\alpha(t)+1}}(60-t). \tag{42}$$

In Fig. 1 we plot $\alpha_{\text{in}}(t)$ and compare it with $\alpha_{\text{out}}(t)$. It will be seen that they agree very well for $t<0$. But

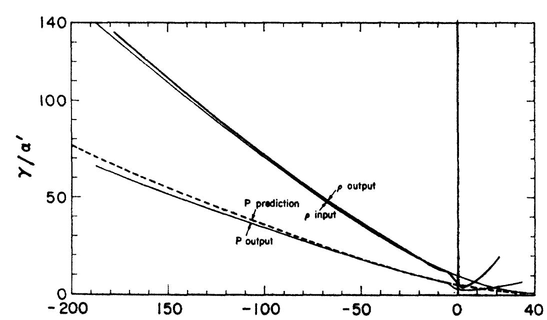

FIG. 2. γ/α' for the ρ and P trajectories. The prediction for P is based on Eqs. (43) and (44).

above $t=0$ they begin to diverge, and in fact $\text{Re}[D_\alpha(s)]$ ceases to have zeros for $\alpha>0.85$. In Fig. 2 we show $(\gamma/\alpha')_{\text{in}}$ and compare it with $N_\alpha/\text{Re}D_\alpha'$ output. Again very good agreement is found except near $t=0$, where the output diverges considerably from our smooth input curve. Figure 3 gives the values of γ_{in} and γ_{out} corresponding to Figs. 1 and 2. Since the potential depends on $\gamma(t)$ and $\alpha(t)$ only for $t<0$, we regard this as a self-consistent solution, but it is clear that our results can not be continued into the physical region.

From (40) the input width of the ρ is 1.95 m_π, or about 2.5 times the generally accepted experimental width. It is not really surprising that we require a larger width, because we have not included the forces from other trajectories, but the magnitude of the discrepancy is a little disturbing. In Fig. 4 we plot the partial-wave cross section for $l=1$. Despite the absence of a zero of $\text{Re}[D_1(s)]$ there is a peak at $s^{1/2}=5.8\ m_\pi$ ($m_\rho=5.4\ m_\pi$), but its full width at half-maximum is $\approx 5\ m_\pi$. Our intercept $\alpha(0)=0.75$ is rather higher than experiment indicates (≈ 0.5),[12] but we have not been able to produce self-consistent trajectories with sufficient slope to pass from 0.5 at $t=0$ to 1.0 at $t=28$. This is possible with more rapidly varying residue functions, but such residues can not be made self-consistent.

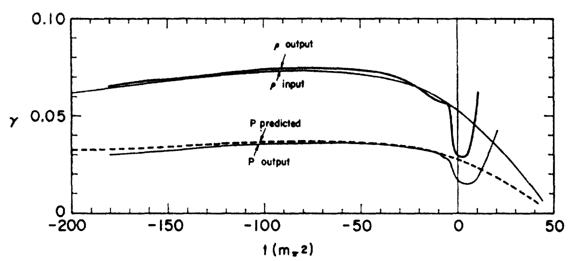

FIG. 3. The residue function γ for the ρ and P trajectories.

VI. THE $I=0$ CHANNEL

Though we have not included the Pomeranchon force, we do of course obtain a trajectory in the $I=0$ channel (A^+), the principal difference from $I=1$ being that crossing matrix element $\beta^{II'}$ is now 1 instead of $\frac{1}{2}$. In fact, with the neglect of the terms mentioned as being small at the end of Sec. II, this is the only difference. Figures 1–3 also include the results for this $I=0$ output. It will be seen that $\alpha(0)$ is slightly greater than 1, the unitarity limit, but this is not surprising in view of the large ρ width we have used. There is no sign of a secondary P' trajectory.

The P trajectory is almost exactly parallel to the ρ,

$$\alpha_P(t)\approx\alpha_\rho(t)+0.320. \tag{43}$$

Using this expression for α_P, we have compared, in Figs. 2 and 3, the output values of γ/α' and γ with the prediction of (39). Remembering the crossing matrix

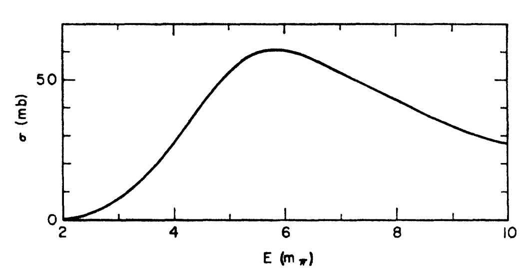

FIG. 4. The $I=1$, p-wave cross section in millibarns versus energy.

[12] R. J. N. Phillips and W. Rarita, Phys. Rev. 139, B1336 (1965).

element, we have

$$\gamma_P(t)=2c\alpha_P'(t)(\bar{t}-t)\frac{Q_{\alpha_P(t)}[1+56/(\bar{t}-4)]}{[(\bar{t}-4)/4]^{\alpha_P(t)+1}}. \quad (44)$$

It will be seen that the prediction is well satisfied except for $t\geqslant 0$.

$\mathrm{Re}\alpha(t)$ has its maximum at $t\approx 20\, m_\pi^2$, though we have not traced the fall of $\mathrm{Re}\alpha(t)$ in Fig. 1, since, because of the large imaginary parts of α and D, it is not correct to identify the second zero of $\mathrm{Re}D$ with the returning trajectory. From (33) we can see that if $\rho_\alpha(t)$ has its main weight in the upper part of the strip one would not expect this maximum to occur for $t<s_1/2$. Our present calculation appears to emphasize the region of the double spectral function just above threshold, so that our results cease to be correct as we enter the resonance region.

We conclude that it may be possible to "bootstrap" trajectories with some hope of obtaining the physical parameters for $t<0$, when all the trajectories are included, but there is no sign that we shall be able to obtain the correct particle masses and widths. It is likely that the presence of competing channels is important for higher angular momenta, and this possibility is being examined.[13] Also it may be necessary to iterate the potential[14] in order to obtain a better approximation to the double spectral functions.

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[13] Shu-Yuan Chiu (private communication).

[14] This procedure has been discussed by B. H. Bransden *et al.*, Nuovo Cimento **30**, 207 (1963), whose results are very encouraging.

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Polarization of a Decay Particle in a Two-Step Process: Application to $K^-+p \rightarrow \pi^0+\Sigma^0$, $\Sigma^0 \rightarrow \gamma+\Lambda$†‡

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The phenomenology of two-step processes of the type $A+B\rightarrow C+D$, $D\rightarrow E+F$ is studied for the particular case when among the final particles only F is observed. Formulas convenient for the computation of the polarization of F in terms of the parameters describing the production process are presented, and the connection between the polarization of F and that of D, when D is not observed, is clarified. Numerical results are obtained for the angular dependence of the Λ polarization in the process $K^-+p\rightarrow\pi^0+\Sigma^0$, $\Sigma^0\rightarrow\gamma+\Lambda$ at a variety of incident energies.

1. INTRODUCTION

PARTICLES with spin are frequently polarized when produced in elementary-particle reactions. As is well known, measurement of this polarization provides restrictions on the values of parameters, e.g., phase shifts, used to describe the matrix element for the production process. If the particle is unstable, the polarization may often be measured from the angular distribution of the decay products and sometimes from the polarization of one of the decay particles.

Special circumstances prevail if the produced particle is the Σ^0 hyperon, e.g., in the reaction

$$K^-+p\rightarrow\pi^0+\Sigma^0. \quad (1.1)$$

The electromagnetic decay of the Σ^0, via

$$\Sigma^0\rightarrow\gamma+\Lambda, \quad (1.2)$$

involves two more neutral particles, and of the three final particles π^0, γ, and Λ, usually only the Λ is de-

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